Math 210 - Final (Fall 2010)

T. Tlas

1. (15 points) Prove that if $K \subset \mathbb{R}$ is compact and $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f(K)$ is compact as well.

2. (7 points each) Consider the following:

i- Prove that the series $\sum_{n=0}^{\infty} \frac{\sin(nx)}{2^n}$ converges for every x.

ii- Let $f(x) = \sum_{n=0}^{\infty} \frac{\sin(nx)}{2^n}$. Prove that $f(x)$ is continuous.

iii- Prove that $\int_{-1}^{1} f(x)\,dx = 0$.

3. (15 points) Suppose $f(x)$ is a real-valued continuous function on $[0,1]$. Assume that $\int_0^1 x^n f(x)\,dx = 0$, $n = 0, 1, 2, \ldots$. Prove that $f(x) = 0 \ \ \forall x \in [0,1]$.

4. (15 points) Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ and suppose L is a real number. Assume that every subsequence of $\{a_n\}_{n=1}^{\infty}$ has a sub-subsequence which converges to L. Prove that $\{a_n\}_{n=1}^{\infty}$ converges to L.

5. (17 points) Prove that $\frac{x}{\frac{\pi}{2}} \leq \sin(x) \leq x$ for all $x \in [0, \frac{\pi}{2}]$.

6. (17 points) Suppose $g(x)$ is Riemann integrable on $[a,b]$ and $g(x) \geq 0$. Assume $f(x)$ is continuous on $[a,b]$. Prove that there exists a $c \in [a,b]$ such that $\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$.